Mar 2

E.g. $f(x) = |x| = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x \leq 0 \end{cases}$

f is continuous,
differentiable except at $x = 0$

$f' = \begin{cases} 1 & \text{when } x > 0 \ (+) \\ -1 & \text{when } x < 0 \ (-) \\ \text{undefined} & \text{when } x = 0 \end{cases}$ ← changes sign at $x = 0$

$x = 0$ ← critical point.

↓ f has a local extremum at $x = 0$ $(-) \to (+)$

$x < 0$ | $x = 0$ | $x > 0$

$f' < 0$ | $f' > 0$

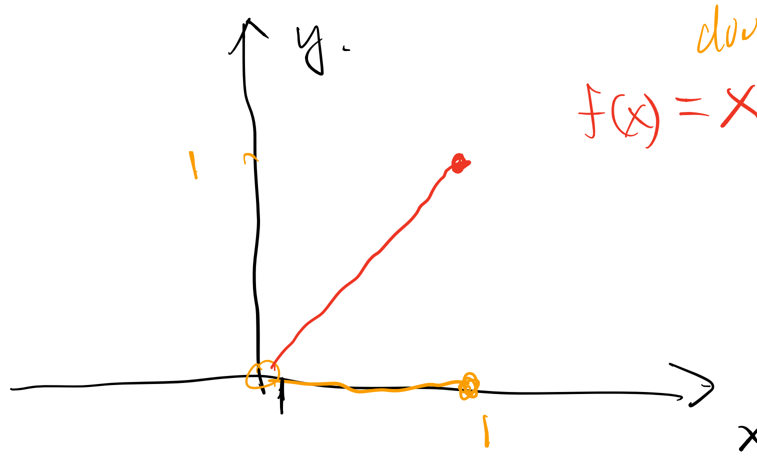
f is strictly decreasing | f. strictly increasing

E.g. Consider $f : [0, 1] \to \mathbb{R}$

$f(x) = x$

the extremal value thm doesn't apply

$f(x) = x$



f has a local minimum at $x = 0$

f has an absolute maximum at $x=1$.

the maximal value of f is 1.

$\left(\quad f(x) \leq f(1) = 1 \quad \right)$.

f does not have an absolute minimum

[an absolute min. occurs at $c \in (0,1]$

if $f(c) \leq f(x) \quad \forall x$)

such c does not exist. b/c

0 is not in the domain of f.

$f(x)$ can be arbitrarily close to 0

but can never be 0.

0 is the largest possible value y

s.t. $f(x) \geq y \quad \forall x \in (0,1]$

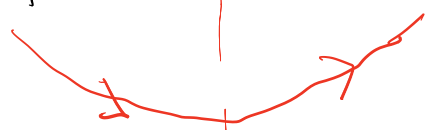
but $f(x) \neq 0$ ~~for~~ any $x \in (0,1]$

<u>E.g.</u> $f(x) = x^2+1$

$f' = 2x = 0$ when $x = 0$

Critical pts of f: $x=0$

f can have a local extrema at $x=0$



$f'<0$ $\quad f'=0 \quad$ $f'>0$

x

$x<0$ $\quad 0 \quad$ $x>0$

f decreasing $\quad$ f increasing

f' changes sign at $x=0$. $\hookrightarrow$ f has a local extremum at $x=0$

$(-) \to (+)$

$\hookrightarrow$ f has a local minimum at $x=0$

E.g.,

$$f(x) = x^3$$

$f' = 3x^2 = 0$ when $x=0$

Critical pts of f: $\quad$ ~~x~~ $x=0$

candidate for local extrema.

$f'>0$ $\qquad$ $f'>0$

$x<0$ $\quad x=0 \quad$ $x>0$

f strictly increasing $\qquad$ strictly increasing

f not a local. extremum.

b/c $f' > 0$ both

when $x > 0$

and when $x < 0$

when $x < -1$
then $f' > 0$. f' is strictly increasing on $(-\infty, -1)$

Procedure to determine intervals of increase/decrease of f

Find all critical pts of f

1. Find all c such that $f'(c) = 0$ or $f'(c)$ is undefined. Divide the line into several intervals.
2. For each intervals (a, b) obtained in the previous step.
   (a) If $f'(x) > 0$, $f(x)$ is a strictly increasing function ($\uparrow$) on (a, b).
   (b) If $f'(x) < 0$, $f(x)$ is a decreasing function ($\downarrow$) on (a, b).

Example 6.2.2. Find the intervals in which the function

$$f(x) = 2x^3 + 3x^2 - 12x - 7$$

is strictly increasing/strictly decreasing.

Solution.

$6(x^2+x-2)$

$$f'(x) = 6x^2 + 6x - 12 = 6(x + 2)(x - 1) = 0 \quad \Rightarrow x = -2, 1.$$

So we have 3 intervals: $(-\infty, -2)$, $(-2, 1)$, $(1, \infty)$.

In $(-\infty, -1)$, $x + 1 < 0, x - 1 < 0$, so $f'(x) > 0$.
In $(-1, 1)$, $x + 1 > 0, x - 1 < 0$, so $f'(x) < 0$.
In $(1, +\infty)$, $x + 1 > 0, x - 1 > 0$, so $f'(x) > 0$.

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$
monotonicity	$\uparrow$		$\downarrow$		$\uparrow$

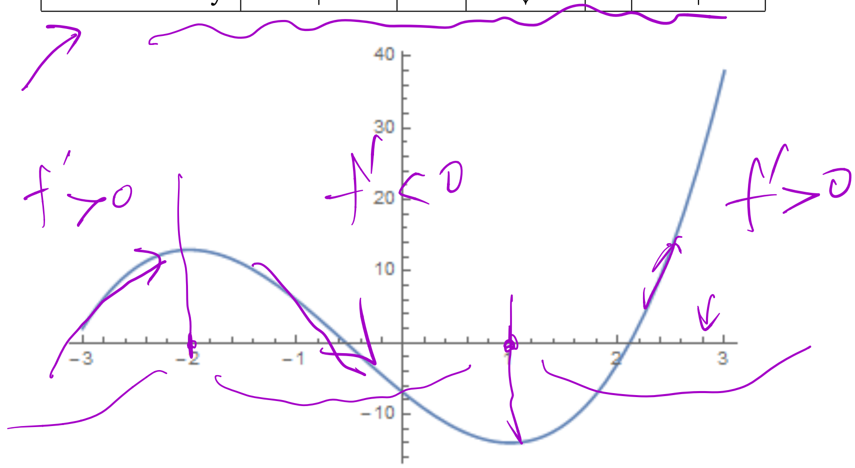


Figure 6.1: $y = 2x^3 + 3x^2 - 12x - 7$

■

Exercise 6.2.1. Find the intervals of strict increase and strict decrease of the function

$$f(x) = x^7 - 2x^5 + x^3.$$

critical pts of f

Solution.

$$f'(x) = 7x^6 - 10x^4 + 3x^2 = x^2(7x^4 - 10x^2 + 3) = 0 \quad \Rightarrow \quad x = 0, \pm 1 \text{ and } \pm\sqrt{\frac{3}{7}} \approx \pm 0.654654.$$

$(7x^2-3)(x^2-1)$

x	$(-\infty, -1)$	$(-1, -\sqrt{\frac{3}{7}})$	$(-\sqrt{\frac{3}{7}}, 0)$	$(0, \sqrt{\frac{3}{7}})$	$(\sqrt{\frac{3}{7}}, 1)$	$(1, +\infty)$
$f'(x)$	$+$	$-$	$+$	$+$	$-$	$+$
monotonicity	$\uparrow$	$\downarrow$	$\uparrow$	$\uparrow$	$\downarrow$	$\uparrow$

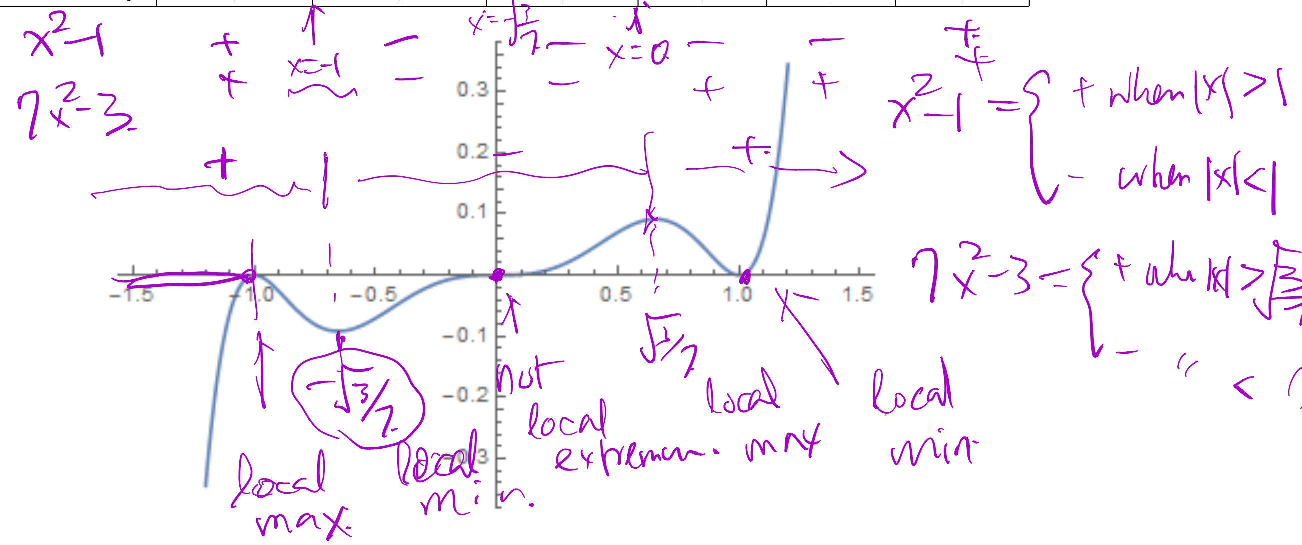


Figure 6.2: $y = x^7 - 2x^5 + x^3$

■

Definition 6.2.2. Let $f(x)$ be a real-valued function defined on (a, b). A number $c \in (a, b)$ is called a critical point of f if $f'(c) = 0$ or $f'(c)$ does not exist.
The corresponding value $f(c)$ is called a critical value for $f(x)$.

Remark. The notion of critical points applies to more general functions, e.g. real functions of several variables, complex functions etc. A critical point always lies in the domain of the function. In the special case of real-valued functions of a single real variable, a critical point is a real number; therefore it is also called a *critical number*. Let $f(x)$ be a real-valued function of a single real variable, and $c \in \mathbb{R}$ be a critical point of f. Let $C \subset \mathbb{R}^2$ be the graph of f in the $x-y$ plane. The point $(c, f(c)) \in C$ is a critical point of the function $\pi_y : C \to \mathbb{R}$ given by $(x, y) \mapsto y$.

Example 6.2.3. Find the intervals of increase/decrease

$$f(x) = |x|. = \begin{cases} x & \text{when } x \geq 0 \\ -x & \text{when } x \leq 0 \end{cases}$$

We have proved

$$f'(x) = \begin{cases} -1, & x < 0, \\ \text{does not exist}, & x = 0, \\ 1, & x > 0. \end{cases}$$

critical pt of f.

$\Rightarrow$ critical number: $x = 0$; corresponding critical value: 0

x	$(-\infty, 0)$	0	$(0, +\infty)$
$f'(x)$	$-$	0	$+$
monotonicity	$\downarrow$		$\uparrow$

Example 6.2.4. $f(x) = x^4 - 4x^3$. Find all critical points and increasing & decreasing intervals.

Solution.

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) = 0 \quad \Rightarrow \quad x = 0, 3.$$

critical points: $x = 0, 3$

corresponding critical values: $f(0) = 0, f(3) = -27$

not local extremum

x	$(-\infty, 0)$	0	$(0, 3)$	3	$(3, +\infty)$
$f'(x)$	$-$	0	$-$	0	$+$
monotonicity	$\downarrow$		$\downarrow$		$\uparrow$

local min. f' $(-) \to (+)$

signs don't change

critical points

∎

6.2.2 Maxima & Minima of Functions

Definition 6.2.3. Let $f(x)$ be a real-valued function with domain I. We say

1. $f(x)$ has a **relative maximum (or local maximum)** at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$ near c.

f has a relative (local) minimum at $x = c$ if $f(c) \leq f(x)$ $\forall x \in I$. where I is an open interval containing c.

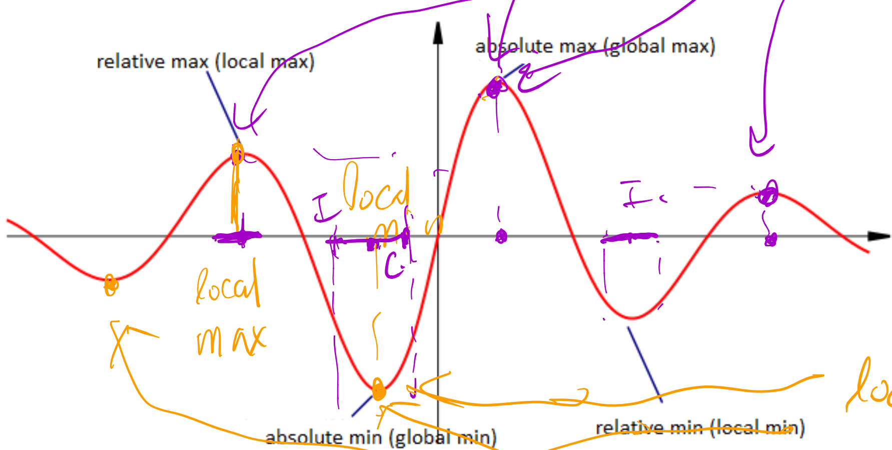
2. $f(x)$ has a **global maximum (or absolute maximum)** at $x = c$ if $f(c) \geq f(x)$ for all $x \in I$.

Similar definition for relative/global minimum.

absolute max

Both maximum and minimum are called an extremum.

local max



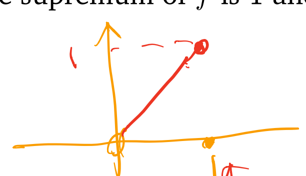
local min.

absolute min.

Remark. Global extremum $\Rightarrow$ Local extremum
But Global extremum $\not\Leftarrow$ Local extremum

Remark. There is some confusion in the literature regarding whether a (local or global) maximum/minimum of a function refers to an element in the domain or its corresponding value (in the range). For most literature, *the* (absolute) maximum of a real function $f(x)$ refers to the value: *$M \in \mathbb{R}$ is said to be the (absolute) maximum if there exists an element c in the domain D of f such that $f(x) \leq f(c)$ $\forall x \in D$.* To be clear, say that M is an (absolute) maximum *value* of f; and f *attains* its (absolute) maximum *at* c. Say e.g. f has local maxima at $x_1, x_2, \ldots \in D$, with corresponding values $f(x_1), f(x_2), \ldots$. Similarly for the notions of (absolute/local) minimum.

Remark. Absolute maxima/minima may not exist. Consider the e.g. the function $f : (0, 1] \to \mathbb{R}$ given by $f(x) = x$. This f has an absolute maximum but has no absolute minimum. A general notion is *supremum/infimum*. In the above example, the supremum of f is 1 and its infimum is 0.



Question I: How to find relative extrema?

Theorem 6.2.2 (First Derivative Test: Relative Extrema)**.**

Let $f(x)$ be a continuous function which is differentiable where $x \neq c$. Then

f has an absolute max at $x = 1$.

1. *$f(x)$ attains a relative maximum at $x = c$ if near the point c,*

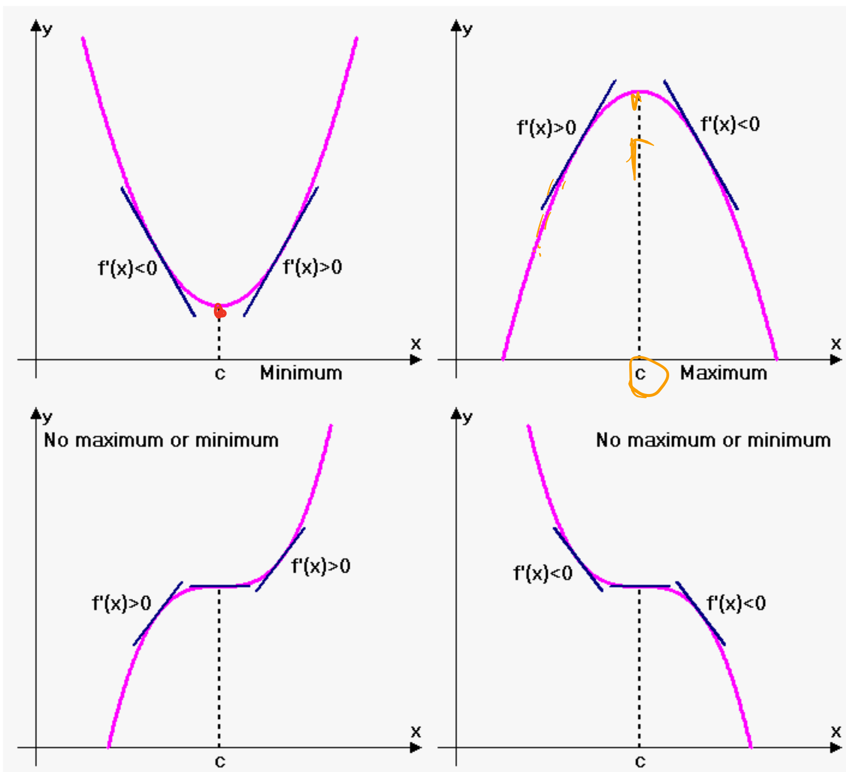
$$f'(x) > 0 \quad \text{for } x < c; \qquad f'(x) < 0 \quad \text{for } x > c.$$

(max value of f is 1)

2. *$f(x)$ attains a relative minimum at $x = c$ if near the point c,*

$$f'(x) < 0 \quad \text{for } x < c; \quad f'(x) > 0 \quad \text{for } x > c.$$

3. *$f(x)$ attains no relative extremum at $x = c$ if near the point c $f'(x)$ has the same sign on two sides of c.*



Property	Sign of $f'(x)$ to the left of c	Sign of $f'(x)$ to the right of c
Relative maximum	$+$	$-$
Relative minimum	$-$	$+$
Not a relative extremum	$+$	$+$
Not a relative extremum	$-$	$-$

Theorem 6.2.3. *Let $c \in (a, b)$ and let f be a continuous function on (a, b) such that f' exists and is continuous on $(a, b)\backslash\{c\}$. Then f attains a relative extremum at $x = c$ $\Rightarrow$ c is a critical number, i.e. $f'(c) = 0$ or $f'(c)$ does not exist.*

Remark. f attains a relative extremum at $x = c$ $\not\Leftarrow$ c is a critical number.
For example, $f(x) = x^3$, $f'(x) = 3x^2$, so $x = 0$ is a critical number. But $f'(x) > 0$ on two sides of $x = 0$, so f does not have a relative extremum at 0.

Example 6.2.5. Let

$$f(x) = 2x^3 + 3x^2 - 12x - 7.$$

Find all its relative maxima and relative minima.

Solution. Refer to the answer of Example 6.2.2, $f'(x) = 6x^2 + 6x - 12$. The critical numbers are solutions of $f'(x) = 0$, i.e $x = -2$ and $x = 1$.

x	$(-\infty, -2)$	-2	$(-2, 1)$	1	$(1, +\infty)$
$f'(x)$	$+$	0	$-$	0	$+$

(point where a relative maximum occurs, corresponding value): $(-2, f(-2)) = (-2, 13)$
(point where a relative minimum occurs, corresponding value): $(1, f(1)) = (1, 14)$

■

Example 6.2.6.

1. For Example 6.2.3 $f(x) = |x|$.
   One critical number: $x = 0$, One relative minimum at 0, with corresponding value 0.
2. For example 6.2.4 $f(x) = x^4 - 4x^3$. $f' = 4x^3 - 12x^2 = 4x^2(x-3) = 0$ when $x=0$ or 3
   critical numbers: $x = 0, 3$, one relative minimum at 3, with corresponding value -27.

Exercise 6.2.2. Let

$$f(x) = x^7 - 2x^5 + x^3.$$

(see Exercise 6.2.1) Find all relative maxima and relative minima of f.

Answer:

$$\text{(point where a relative maximum occurs, corresponding value)}:$$
$$(-1, f(-1)) = (-1, 0); (\sqrt{\tfrac{3}{7}}, f(\sqrt{\tfrac{3}{7}}) \approx (0.655, 0.092)$$
$$\text{(point where a relative minimum occurs, corresponding value)}:$$
$$(-\sqrt{\tfrac{3}{7}}, f(-\sqrt{\tfrac{3}{7}})) \approx (-0.655, -0.092); (1, f(1)) = (1, 0).$$

Note that f has no relative extremum at 0.

Question II: How to find absolute Max/Min?

Theorem 6.2.4. *Suppose $f : [a, b] \to \mathbf{R}$ is a continuous function, then the absolute maximum point and absolute minimum point exist for the graph of f (Theorem 3.2.2 Extreme Value Theorem).*

Remark. Note that the preceding theorem applies only when the domain of f is *closed*!

finite interval

Procedures to find absolute max/min of continuous function f on $[a, b]$

1. Find all the critical numbers $c_1, c_2, \ldots,$ in (a, b).
2. Compute the values $f(a), f(b), f(c_1), f(c_2), \ldots,$
   The maximum value corresponds to the absolute max.
   The minimum value corresponds to the absolute min.

Example 6.2.7. Find the absolute maximum and absolute minimum of $f(x) = x^5 - 80x$ on $[-3, 4]$.

Solution. Since $f(x)$ is continuous on $[-3, 4]$, the absolute max/min can be reached by extreme value theorem.

$$f'(x) = 5x^4 - 80 = 0 \quad \Rightarrow \quad x = -2, 2.$$

Compute

$$f(-2) = 128, \quad f(2) = -128,$$
$$f(-3) = -3, \quad f(4) = 704.$$

The absolute minimum is -128, attained at $x = 2$; the absolute maximum is 704, attained at $x = 4$.

abs. max.

not abs. max

■

Exercise: Find the point when abs min of f is attained.

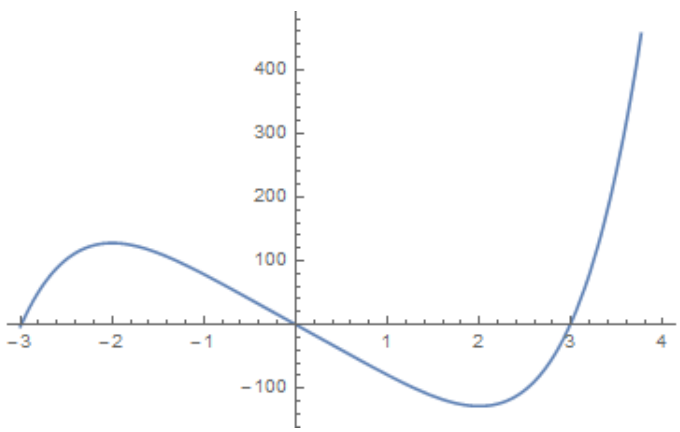


Figure 6.3: $y = x^5 - 80x$ over $[-3, 4]$